Vectors Cheat Sheet

Edexcel Core Pure 1

Equation of a line in three dimensions

You need to know how to express the equation of a straight line in three dimensions in both vector and cartesian form. The equation of a straight line that passes through the position vector $\boldsymbol{a}$ and is parallel to the vector $\boldsymbol{b}$ is written as:

- $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ in vector form. — r is a general point and λ is a scalar parameter.
- $\dfrac{x-a_1}{b_1} = \dfrac{y-a_2}{b_2} = \dfrac{z-a_3}{b_3}$ in cartesian form. — where $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Example 1: Find a vector equation of the straight line which passes through the points A and B, with position vectors $4\boldsymbol{i}+5\boldsymbol{j}-\boldsymbol{k}$ *and* $6\boldsymbol{i}+3\boldsymbol{j}+2\boldsymbol{k}$ respectively.

We know the line passes through both A and B so we can use either as our a.	$r = \boldsymbol{a} + \lambda\boldsymbol{b}$ $\boldsymbol{a} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$
To find the direction vector $\boldsymbol{b}$, subtract the position vectors.	$\boldsymbol{b} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$
So, the vector equation of the line is:	$\therefore \boldsymbol{r} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$

Example 2: With respect to the fixed origin O, the line l is given by the equation $\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. Prove that the cartesian form of l is given by $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$.

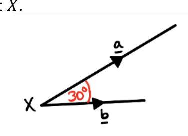
Write the general point $\boldsymbol{r}$ as $\begin{pmatrix} x \\ y \\ x \end{pmatrix}$.	$\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$
Equate each component.	$x = a_1 + \lambda b_1$ $y = a_2 + \lambda b_2$ $z = a_3 + \lambda b_3$
Rearrange each equation for λ.	$\lambda = \frac{x-a_1}{b_1},\ \lambda = \frac{y-a_2}{b_2},\ \lambda = \frac{z-a_3}{b_1}$
Equating each equation gives us the result.	$\therefore \lambda = \frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$

Finding the angle between two vectors

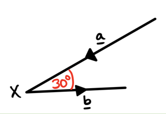
Before we discuss the scalar product, which plays a large role in this chapter, it is important that you know how to correctly find the angle between two vectors.

- The angle between two vectors **a** and **b** is the angle between them when both are pointed away from their point of intersection.

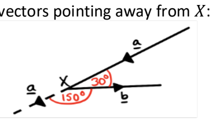
In the following diagram, the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, since both vectors point away from the point X.



In the below diagram, it is tempting to deduce that the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$. But this is in fact not true because the vectors are not both pointing away from the point X.



We can re-draw the diagram to show both vectors pointing away from X:



This new diagram shows us that the actual angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$.

Scalar product (dot product)

The scalar product is a function which takes two vectors and outputs a number. The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is written as $a \cdot b$. You will need to use the scalar product to find angles between two vectors. It is defined as:

- $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ — To find angles, rearrange the formula into: $\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$

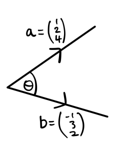
where $\boldsymbol{a} \cdot \boldsymbol{b}$ represents the scalar product of the vectors a and b. In order to compute the scalar product of two vectors, you can use the following fact:

- If $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The following fact is also very important:

- The non-zero vectors a and b are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Example 3: Find the angle θ between vectors **a** and **b**.



$\boldsymbol{a} \cdot \boldsymbol{b} = 1(-1) + 2(3) + 4(2) = 13$
$\lvert\boldsymbol{a}\rvert = \sqrt{1^2+2^2+4^2} = \sqrt{21}$
$\lvert\boldsymbol{b}\rvert = \sqrt{1^2+3^2+2^2} = \sqrt{14}$
$\therefore \cos\theta = \frac{13}{(\sqrt{21})(\sqrt{14})} = 0.758$
$\Rightarrow \theta = cos^{-1}(0.758) = 40.7°$

Equation of a plane in three dimensions

You also need to be able to express the equation of a plane in both vector and cartesian form. The direction of a plane is described by a normal vector, often denoted as $\boldsymbol{n}$. This is simply a vector that is perpendicular to the plane.

- $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$ — $\boldsymbol{a}$ is a point that lies on the plane and $\boldsymbol{b}$ and $\boldsymbol{c}$ are both vectors that lie on the plane. λ and μ are scalar parameters.
- $n_1x + n_2y + n_3z = d$

The cartesian form can be simplified using the scalar product: — where $\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ and $\boldsymbol{d} = \boldsymbol{a} \cdot \boldsymbol{n}$

- $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

$\boldsymbol{r}$ is a general point. This form is often more useful for the problems you will encounter in this chapter. To convert back into the form $n_1x + n_2y + n_3z = d$, you need to replace $\boldsymbol{r}$ with $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and expand the scalar product.

Finding angles between lines and planes

- The acute angle θ between two intersecting lines is given by $\cos\theta = \left|\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right|$.
- The acute angle θ between the line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ and the plane $\boldsymbol{r} \cdot \boldsymbol{n} = k$ is given by $\sin\theta = \left|\frac{\boldsymbol{b}\cdot\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$.
- The acute angle θ between the plane $\boldsymbol{r} \cdot \boldsymbol{n_1} = k_1$ and the plane $\boldsymbol{r} \cdot \boldsymbol{n_2} = k_2$ is given by $\cos\theta = \left|\frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right|$.

These formulas will give you an acute angle. If you instead wish to find the obtuse angle, subtract your answer from 180.

Example 4: The lines l_1 and l_2 have vector equations $\boldsymbol{r} = (2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) + t(3\boldsymbol{i}-8\boldsymbol{j}-\boldsymbol{k})$ and $\boldsymbol{r} = (7\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k}) + s(2\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})$ respectively. Given that l_1 and l_2 intersect, find the size of the acute angle between the lines to one decimal place.

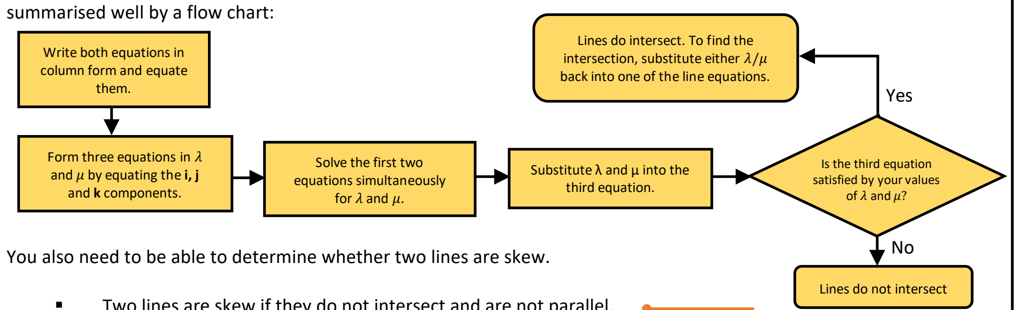
Use the direction vectors and take the scalar product.	$\begin{pmatrix} 3 \\ -8 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = 3(2) - 8(2) - 1(3) = -13$
Find the magnitude of the direction vectors.	$\lvert l_1\rvert = \sqrt{(3)^2+(-8)^2+(-1)^2} = \sqrt{74}$ $\lvert l_2\rvert = \sqrt{(2)^2+(2)^2+(3)^2} = \sqrt{17}$
Use $\cos\theta = \left\lvert\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert}\right\rvert$	$\cos\theta = \left\lvert\frac{-13}{(\sqrt{74})(\sqrt{17})}\right\rvert = 0.367$ $\therefore \theta = cos^{-1}(0.367) = 68.5°$

Example 5: Find the acute angle between the line with equation $\boldsymbol{r} = (2\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k}) + t(3\boldsymbol{i}+4\boldsymbol{j}-12\boldsymbol{k})$ and the plane with equation $\boldsymbol{r} \cdot (2\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}) = 2$.

Use the direction vector of the line and the normal vector of the plane and take the scalar product.	$\begin{pmatrix} 3 \\ 4 \\ -12 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 3(2) + 4(-2) - 12(-1) = 10$
Find the magnitude of the direction vectors:	$\lvert l_1\rvert = \sqrt{(3)^2+(4)^2+(-12)^2} = 13$ $\lvert l_2\rvert = \sqrt{(2)^2+(-2)^2+(-1)^2} = 3$
Use $\sin\theta = \left\lvert\frac{\boldsymbol{b}\cdot\boldsymbol{n}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{n}\rvert}\right\rvert$	$\sin\theta = \left\lvert\frac{10}{(13)(3)}\right\rvert = 0.256$ $\therefore \theta = sin^{-1}(0.256) = 14.9°$

Finding whether two lines intersect

You need to be able to determine whether two lines meet or not and find their intersection if they do. The procedure is summarised well by a flow chart:



You also need to be able to determine whether two lines are skew.

- Two lines are skew if they do not intersect and are not parallel.

Example 6: The lines l_1 and l_2 have vector equations $\boldsymbol{r} = (2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}) + t(3\boldsymbol{i}-8\boldsymbol{j}-\boldsymbol{k})$ and $\boldsymbol{r} = (7\boldsymbol{i}+4\boldsymbol{j}+\boldsymbol{k}) + s(2\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})$ respectively. Show that the two lines do not intersect.

Write both lines in column form and equate.	$\begin{pmatrix} 2+3t \\ 1-8t \\ 1-t \end{pmatrix} = \begin{pmatrix} 7+2s \\ 4+2s \\ 1+3s \end{pmatrix}$
Form three equations.	$2+3t = 7+2s$ [1] $1-8t = 4+2s$ [2] $1-t = 1+3s$ [3]
Solve the first two equations simultaneously.	$t = \frac{2}{11},\quad s = -\frac{49}{22}$
Substitute these values of s and t into equation [3].	$LHS = 1 - \frac{2}{11} = \frac{9}{11}$ $RHS = 1 + 3\left(-\frac{49}{22}\right) = -\frac{125}{22}$
Since the third equation is not satisfied by our solutions, we can conclude the lines don't intersect.	$LHS \neq RHS$, so, the three equations are not consistent. Hence the lines do not intersect.

Example 7: The lines l_1 and l_2 have equations $\frac{x-2}{4} = \frac{y+3}{2} = z - 1$ and $\frac{x+1}{5} = \frac{y}{4} = \frac{z-4}{-2}$ respectively. Prove that l_1 and l_2 are skew.

Firstly, we need to show that these lines don't intersect. Write both lines in vector column form and equate.	$\begin{pmatrix} 2+4\lambda \\ -3+2\lambda \\ 1+\lambda \end{pmatrix} = \begin{pmatrix} -1+5\mu \\ 4\mu \\ 4-2\mu \end{pmatrix}$
Form three equations.	$2+4\lambda = -1+5\mu$ [1] $-3+2\lambda = 4\mu$ [2] $1+\lambda = 4-2\mu$ [3]
Solve the first two equations simultaneously.	$\lambda = -\frac{9}{2},\quad \mu = -3$
Substitute these values of s and t into equation [3].	$LHS = 1 - \frac{9}{2} = -\frac{7}{2}$ $RHS = 4 - 2(-3) = 10$
Since the third equation is not satisfied by our solutions, we can conclude the lines don't intersect.	$LHS \neq RHS$, so the three equations are not consistent. Hence the lines do not intersect.
Now all we need to do to prove these lines are skew is show that the direction vectors are not parallel.	$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \neq k\begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}$ for any choice of k. Hence, they are not parallel. So, the lines are skew.

Finding the intersection between a line and a plane

- To find the intersection between a line and a plane, first express the plane in the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{k}$ and replace the general point **r** with the vector equation of the line. Then solve the resultant equation.

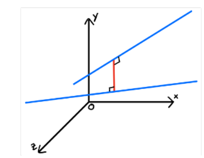
Example 8: Find the coordinates of the point of intersection of the line l and the plane Π where l has the equation $\boldsymbol{r} = -\boldsymbol{i}+\boldsymbol{j}-5\boldsymbol{k}+\lambda(\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k})$ and Π has equation $\boldsymbol{r}\cdot(\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k}) = 4$.

First write the equation of the line in vector column form.	Equation of line: $\boldsymbol{r} = \begin{pmatrix} -1+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix}$
Replace $\boldsymbol{r}$ in the equation of the plane with the vector equation of the line.	$\begin{pmatrix} -1+\lambda \\ 1+\lambda \\ -5+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4$
Use the dot product.	$-1+\lambda+(1+\lambda)(2)+(-5+2\lambda)(3) = 4$
Solve the subsequent equation for λ.	$\lambda = 2$
Substitute λ back into the equation of the line to find the point of intersection.	$\therefore$ point of intersection $= \begin{pmatrix} -1+(2) \\ 1+(2) \\ -5+2(2) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

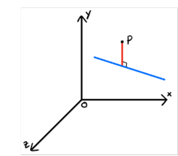
Finding shortest distances and reflections

You need to be able to find the perpendicular distance between two lines, a point and a line as well as a point and a plane. This is also the shortest distance between them. Through the use of examples, we will explain how to find the shortest distance in each of these scenarios. We will also look at how to find the reflection of a point or a line in a plane.

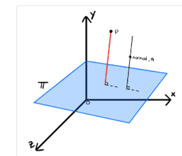
For any two non-intersecting lines, there is a unique line segment AB which is perpendicular to both lines.



The perpendicular from the point P to a line is a line which passes through P and meets the line at a right angle.



The perpendicular from a point P to a plane is a line which passes through P and is parallel to the normal vector, **n**.



Shortest distance between two parallel

Example 9: Find the shortest distance between the lines with equations $\boldsymbol{r} = (\boldsymbol{i}-2\boldsymbol{j}-\boldsymbol{k}) + \lambda(5\boldsymbol{i}+4\boldsymbol{j}+3\boldsymbol{k})$ and $\boldsymbol{r} = (2\boldsymbol{i}+\boldsymbol{k}) + \mu(5\boldsymbol{i}+4\boldsymbol{j}+3\boldsymbol{k})$.

Let A be a general point on the first line and B be a general point on the second. The shortest distance is the length of $\overrightarrow{AB}$ when $\overrightarrow{AB}$ is perpendicular to both l_1 and l_2.	$\overrightarrow{AB} = \begin{pmatrix} 2+5\mu \\ 4\mu \\ 1+3\mu \end{pmatrix} - \begin{pmatrix} 1+5\lambda \\ -2+4\lambda \\ -1+3\lambda \end{pmatrix} = \begin{pmatrix} 1+5(\mu-\lambda) \\ 2+4(\mu-\lambda) \\ 2+3(\mu-\lambda) \end{pmatrix}$
Let $t = (\mu - \lambda)$, then $\overrightarrow{AB}$ becomes:	$= \begin{pmatrix} 1+5t \\ 2+4t \\ 2+3t \end{pmatrix}$
The shortest distance is when $\overrightarrow{AB}$ is perpendicular to both lines, so:	$\begin{pmatrix} 1+5t \\ 2+4t \\ 2+3t \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = 0$
Substitute these values of s and t into equation [3].	$5(1+5t)+4(2+4t)+3(2+3t) = 0$
Rearrange for t.	$t = -0.38$
Substitute t back into $\overrightarrow{AB}$. The required distance will be given by $\lvert\overrightarrow{AB}\rvert$.	$\overrightarrow{AB} = \begin{pmatrix} 1+5(-0.38) \\ 2+4(-0.38) \\ 2+3(-0.38) \end{pmatrix} = \begin{pmatrix} -0.9 \\ 0.48 \\ 0.86 \end{pmatrix}$ $\therefore \lvert\overrightarrow{AB}\rvert = \sqrt{(-0.9)^2+(0.48)^2+(0.86)^2} = \frac{\sqrt{89}}{5\sqrt{2}}$

Shortest distance between two non-parallel

Example 10: Find the shortest distance between the lines l_1 and l_2 with equations $\boldsymbol{r} = (\boldsymbol{i}) + \lambda(\boldsymbol{j}+\boldsymbol{k})$ and $\boldsymbol{r} = (-\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}) + \mu(2\boldsymbol{i}-\boldsymbol{j}-\boldsymbol{k})$ respectively.

Let A be a general point on the first line and B be a general point on the second. The shortest distance is the length of $\overrightarrow{AB}$ when $\overrightarrow{AB}$ is perpendicular to both l_1 and l_2.	$\overrightarrow{AB} = \begin{pmatrix} -1+2\mu \\ 3-2\mu \\ -1-\mu \end{pmatrix} - \begin{pmatrix} 1 \\ \lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} -2+2\mu \\ 3-2\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix}$
Since $\overrightarrow{AB}$ is perpendicular to l_1, $\overrightarrow{AB} \cdot (direction\ vector\ l_1) = 0$.	$\begin{pmatrix} -2+2\mu \\ 3-2\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$
Calculate dot product.	$2-2\mu-2\lambda = 0$ [1]
$\overrightarrow{AB}$ is also perpendicular to l_2, so $\overrightarrow{AB} \cdot (direction\ vector\ l_2) = 0$.	$\begin{pmatrix} -2+2\mu \\ 3-2\mu-\lambda \\ -1-\mu-\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = 0$
Calculate dot product.	$-6+6\mu+2\lambda = 0$ [2]
Solve [1] and [2] simultaneously.	$\mu = 1,\quad \lambda = 0$
Substitute these values of μ and λ back into $\overrightarrow{AB}$. The shortest distance is given by $\lvert\overrightarrow{AB}\rvert$.	$\overrightarrow{AB} = \begin{pmatrix} -2+2(1) \\ 3-2(1)-(0) \\ -1-(1)-(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$ $\therefore \lvert\overrightarrow{AB}\rvert = \sqrt{(0)^2+(2)^2+(-2)^2} = 2\sqrt{2}$

Shortest distance between a point and a

Example 11: The line l has equation $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z+3}{-1}$, and the point A has coordinates $(1, 2, -1)$. Find the shortest distance between A and l.

First rewrite the line in vector form:	$l\colon r = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$
Let B be a general point on l. Then the vector $\overrightarrow{AB}$ is:	$\overrightarrow{OB} = \begin{pmatrix} 1+2t \\ 1-2t \\ -3-t \end{pmatrix}$, so $\overrightarrow{AB} = \begin{pmatrix} 1+2t \\ 1-2t \\ -3-t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2t \\ -1-2t \\ -2-t \end{pmatrix}$
The shortest distance is when $\overrightarrow{AB}$ is perpendicular to l, so $\overrightarrow{AB} \cdot (direction\ vector\ l) = 0$	$\begin{pmatrix} 2t \\ -1-2t \\ -2-t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 0$
Solving the subsequent equation:	$2(2t) - 2(-1-2t) - 1(-2-t) = 0$ $t = -\frac{4}{9}$
Substitute t back into $\overrightarrow{AB}$. The required distance will be given by $\lvert\overrightarrow{AB}\rvert$.	$\overrightarrow{AB} = \begin{pmatrix} 2(-4/9) \\ -1-2(-4/9) \\ -2-(-4/9) \end{pmatrix} = \begin{pmatrix} (-8/9) \\ (-1/9) \\ (-14/9) \end{pmatrix}$ $\therefore \lvert\overrightarrow{AB}\rvert = \sqrt{\left(-\frac{8}{9}\right)^2 + \left(-\frac{1}{9}\right)^2 + \left(-\frac{14}{9}\right)^2} = 1.80$

Shortest distance between a point and a plane

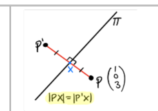
- The perpendicular (shortest) distance from the point (α, β, γ) to the plane $ax + by + cz = d$ is $\dfrac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2+b^2+c^2}}$ — You are given this result in the formula booklet

Example 11: Find the perpendicular distance from the point with coordinates $(3, 2, -1)$ to the plane with equation $2x - 3y + z = 5$.

Use the above formula.	$\frac{\lvert 2(3)-3(2)+1(-1)-5\rvert}{\sqrt{(2)^2+(-3)^2+(1)^2}} = \frac{6}{\sqrt{14}}$

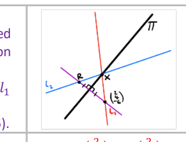
Reflection of a point in a plane

Example 13: The plane Π has equation $-2x + y + z = 5$. The point P has coordinates $(1, 0, 3)$. Find the coordinates of the reflection of the P in Π.

We begin by drawing a diagram. The reflected point is P'. We need to first find the equation of the red line, which is normal to the plane and passes through P.	
Find an equation for the red line.	Equation of red line: $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$
Substitute values into Π equation.	$-2(1-2\lambda) + \lambda + (3+\lambda) = 5$
Solve the subsequent equation.	$1 + 6\lambda = 5$ $\Rightarrow \lambda = \frac{2}{3}$
To find the reflected point, we simply substitute $\lambda = 2(\frac{2}{3})$ into our line. This is because the reflected point P' lies twice as far from P as X.	$\therefore OP' = \begin{pmatrix} 1-2(4/3) \\ (4/3) \\ 3+(4/3) \end{pmatrix} = \begin{pmatrix} -5/3 \\ 4/3 \\ 13/3 \end{pmatrix}$

Reflection of a line in a plane

Example 14: The line l_1 has equation $\frac{x-2}{2} = \frac{y-4}{-2} = \frac{z+6}{1}$. The plane Π has equation $2x - 3y + z = 8$. The line l_2 is a reflection of the line l_1 in Π. Find the equation of l_2.

We begin by drawing a diagram. To find l_2, we need to find two points that lie on l_2. One can be found by finding the intersection of l_1 and Π and the other by reflecting the point (2,4,-6).	
Find the intersection of l_1 and Π.	$l_1\colon \boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Substituting into Π: $\begin{pmatrix} 2+2\lambda \\ 4-2\lambda \\ -6+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 8$
Solve the subsequent equation.	$2(2+2\lambda) - 3(4-2\lambda) + 1(-6+\lambda) = 0$ $\Rightarrow \lambda = 2$
Substitute $\lambda = 2$ into l_1 to find the intersection X.	$\therefore X = (6, 0, -4)$
To find the second point, we can find the reflection of the point $(2, 4, -6)$ in Π. To do this we use the same process as in example 13.	We know that a normal of Π is (2**i**-3**j**+**k**). So, the equation of the line that is normal to the plane and passes through (2, 4, -6) is: $\boldsymbol{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$
Find where this new line intersects Π, and use the solution of λ to find the reflection of (2, 4, -6) which we call R.	$\begin{pmatrix} 2+2\lambda \\ 4-3\lambda \\ -6+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 8$ $\Rightarrow \lambda = \frac{11}{7}$ so $\overrightarrow{OR'} = \begin{pmatrix} 2+2\left(\frac{22}{7}\right) \\ 4-3\left(\frac{22}{7}\right) \\ -6+\left(\frac{22}{7}\right) \end{pmatrix} = \begin{pmatrix} 58/7 \\ -38/7 \\ -20/7 \end{pmatrix}$
To find the direction of the line l_2, find $\overrightarrow{RX}$ or $\overrightarrow{XR}$.	$\overrightarrow{RX} = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 58/7 \\ -38/7 \\ -20/7 \end{pmatrix} = \begin{pmatrix} 16/7 \\ -38/7 \\ 8/7 \end{pmatrix}$
l_2 passes through X (and R).	$l_2\colon \boldsymbol{r} = \begin{pmatrix} 6 \\ 0 \\ -4 \end{pmatrix} + \lambda\begin{pmatrix} 16/7 \\ -38/7 \\ 8/7 \end{pmatrix}$